Day 1: Koopman and Stochastic Systems

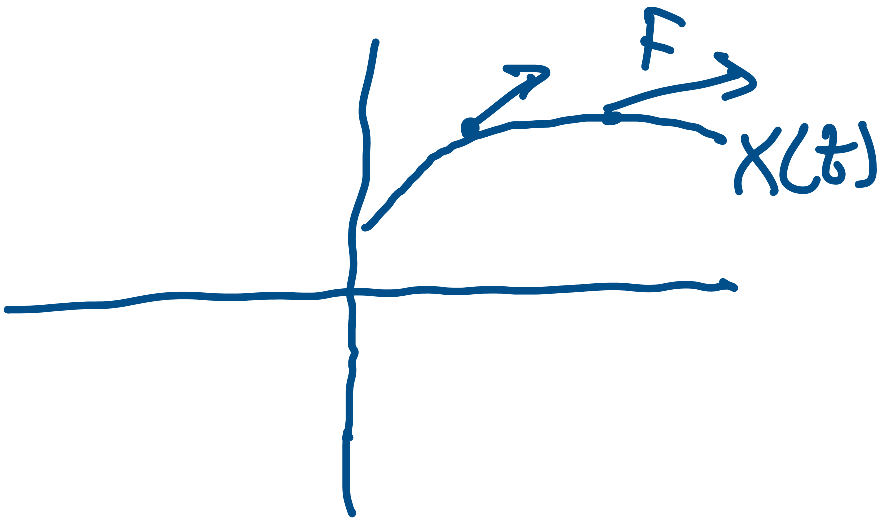
Mittwoch, 18. September 2024 09:01

Ordinary Differential Equation (ODE)

$$(1) \quad \dot{x}(t) = F(x(t)) \quad ; \quad x(0) = x_0$$

$x(t): \mathbb{R} \to \mathbb{R}^d$

Geometric Interpretation:



Linear Approximation for small Δ_t:

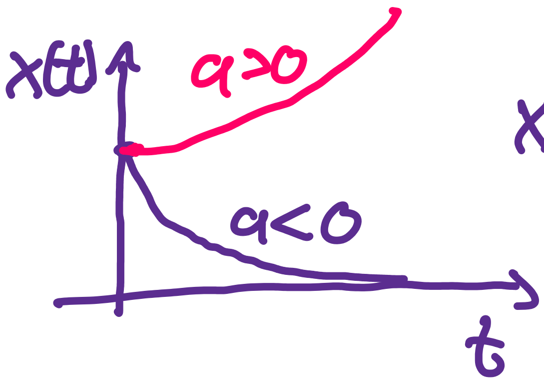
$$x(t+\Delta_t) = x(t) + \Delta_t F(x(t)) + \ldots$$

Euler Scheme:

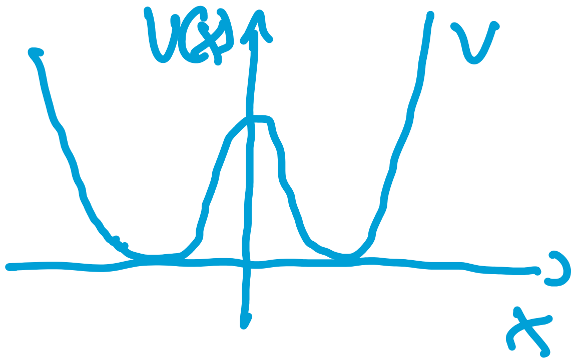
$$x_{k+1} = x_k + \Delta_t F(x_k)$$

Linear ODE: $\dot{x}(t) = a x(t)$, $a \in \mathbb{R}$

$$x(t) = e^{at} x_0 .$$



Gradient-Flow ODE $\dot{x}(t) = -V'(x(t))$



Hamiltonian ODE:

$$\dot{q}(t) = p(t)$$

$$\dot{p}(t) = -\nabla V(q(t))$$

Total Energy $H(q,p) = V(q) + \frac{1}{2} p^2$

is conserved along trajectories of the system.

Evolution of the Dynamics induces the flow map

$\Phi_t x \hat{=}$ "position $X(t)$ if ODE (1) is solved starting from x".

- defines a diffeomorphism on state space $\mathbb{R}^d$.
- <u>semigroup property</u>: $\Phi_{s+t} x = \Phi_s \Phi_t x$

What happens to an <u>observation</u> / <u>measurement</u> function $\phi : \mathbb{R}^d \to \mathbb{R}$?

Define the function
$$v(x,t) = \phi(X_t) = \phi(\Phi_t x).$$

If we view this as an operation on ϕ, this defines a map from functions to functions:

$$[K^t \phi](x) = \phi(\Phi_t x)$$

- this operation is <u>linear</u> on function space!
- it inherits the semigroup property from the flow map:

$$[K^{s+t}\phi](x) = \phi(\Phi_{s+t}x)$$
$$= \phi(\Phi_s \Phi_t(x))$$
$$= [K^s\phi](\Phi_t x)$$
$$= [K^t K^s \phi](x).$$

this is a <u>semigroup</u> of <u>linear</u> operators on function space.

We define the family of <u>Koopman operators</u> as the semigroup of operators K^t.

There is a corresponding <u>linear evolution equation</u> in function space.

Again, for fixed ϕ, define $v(x,t) =$

$$[K^t\phi](x) = \phi(\Phi_t x)$$

$$\frac{\partial}{\partial t} v(x,t) = \nabla\phi(\Phi_t x)\cdot F(\Phi_t x)$$
$$= (F\cdot\nabla)\, v(x,t).$$

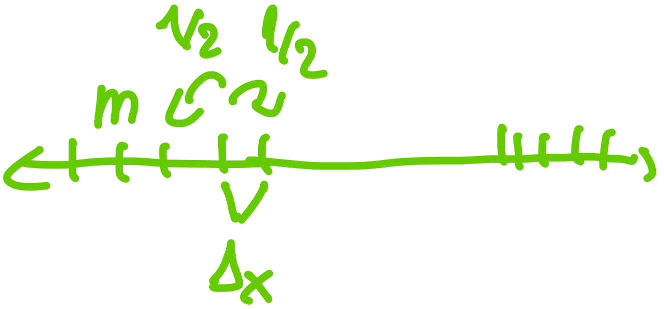
First-order linear PDE for the measurement $\phi(\Phi_t x)$.

- Main Idea of the Koopman approach: discretize the linear time-evolution in function space.

Stochastic Differential Equations

$$X(t+\Delta t) = X(t) + \Delta t\, F(X(t)) + \text{"noise"}$$

Random Walk: discrete jump process on a grid of width Δx, w/ time step Δt:



enumerate time by n.

Probability of meeting the random walker at m after n steps:

$p(m,n)$

Evolution Equation:

$$p(m,n+1)-p(m,n)=\frac{1}{2}\big(p(m+1,n)+p(m-1,n)-2p(m,n)\big).$$

Denote: $\sigma^2=\frac{\Delta x^2}{\Delta t}$

$$\rightarrow \frac{1}{\Delta t}\big(p(m,n+1)-p(m,n)\big)$$

$$=\frac{\sigma^2}{2}\frac{1}{\Delta x^2}\big[p(m+1,n)+p(m-1,n)-2p(m,n)\big].$$