Day 2

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$$\frac{\partial}{\partial t} p(y,t) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} p(y,t)$$

Solution for $p(y,0) = \delta(y-x)$:

$$p(y,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{1}{2\sigma^2 t}(y-x)^2\right)$$

If we call the continuum limit process B_t, the Brownian motion, then its increments must obey

$$B_{t+\Delta t} - B_t \sim \mathcal{N}(0, \sqrt{\Delta t}).$$

We can now use the Brownian increments to drive a stochastic dynamics:

$$X(t+\Delta t) = X(t) + \Delta t\, F(X(t)) + \sigma(B_{t+\Delta t} - B_t)$$

$\rightarrow$ stochastic Euler scheme:

$$X_{k+1} = X_k + \Delta_t F(X_k) + \sigma \sqrt{\Delta_t}\, \mathcal{N}(0,1)$$

We write

$$dX_t = F(X_t)\,dt + \sigma\, dB_t$$

<u>Stochastic Koopman Operator:</u>

Track only average values of measurement functions:

$$[\mathcal{K}^t \phi](x) = \mathbb{E}[\phi(X_t) \mid X_0 = x]$$

$$= \mathbb{E}^x[\phi(X_t)]$$

$$= \int \phi(y) \underbrace{p(x, dy, t)}_{\text{transition kernel}}$$

Semigroup property of transition kernels leads to

$$\mathcal{K}^{s+t} = \mathcal{K}^s \mathcal{K}^t.$$

Evolution Equation for $v(x,t) = E^x[\phi(X_t)]$:

$$\frac{\partial}{\partial t} v(x,t) = \left\{ F \cdot \nabla + \frac{1}{2}\sigma^2 \Delta \right\} v(x,t).$$

→ second-order terms from Ito's Lemma

Learning the Koopman Operator from Data

- fix a finite-dimensional space $\operatorname{span}\{\phi_1, \ldots, \phi_N\}$
- we want to learn the matrix representation of $\mathcal{K}^t$ on this space.

- Assume we have data pairs (x_k, y_k), $1 \le k \le m$, each x_k / y_k are separated by time t.
- Evaluate all basis functions on the data:

$$\underline{\Phi}(X) = \left[\phi_i(x_\ell)\right]_{\substack{1\le i\le N \\ 1\le \ell\le m}} \in \mathbb{R}^{N\times m},$$

$$\underline{\Phi}(Y) = \left[\phi_i(y_\ell)\right] \in \mathbb{R}^{N\times m}.$$

- Now, solve the learning problem

$$K^t = \underset{A\in\mathbb{R}^{N\times N}}{\operatorname{argmin}} \left\| \underline{\Phi}(Y) - A^T \underline{\Phi}(X) \right\|_F^2$$

Analytical solution: $K^t = \underline{\Phi}(X)^{\dagger}\,\underline{\Phi}(Y)$

$$= \big(\underbrace{\underline{\Phi}(X)\,\underline{\Phi}(X)^T}_{\hat{G}}\big)^{-1} \underbrace{\underline{\Phi}(X)\,\underline{\Phi}(Y)^T}_{\hat{A}_t}.$$

Study infinite-sample limits of $\hat{G}$, $\hat{A}_t$:

$$\hat{G}_{ij} = \frac{1}{m} \sum_{\ell=1}^{m} \phi_i(x_\ell)\,\phi_j(x_\ell) \xrightarrow{m\to\infty}$$

$$\xrightarrow{m\to\infty} \mathbb{E}^{\rho}[\phi_i \phi_j] = \int \phi_i(x)\,\phi_j(x)\, d\rho(x)$$

Consider $\hat{A}_t$:

$$\hat{A}_{ij} = \frac{1}{m} \sum_{\ell=1}^{m} \phi_i(x_\ell)\,\phi_j(y_\ell)$$

$$\xrightarrow{m\to\infty} \int \phi_i(x) \left[\int \phi_j(y)\, p(x, dy, t) \right] d\rho(x)$$

$$= \int \phi_i(x)\, K^t \phi_j(x) \qquad d\rho(x)$$

→ <u>Extended Dynamic Mode Decomposition</u>

(EDMD).